An embedding-based distance for temporal graphs

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Problem statement

Temporal graphs

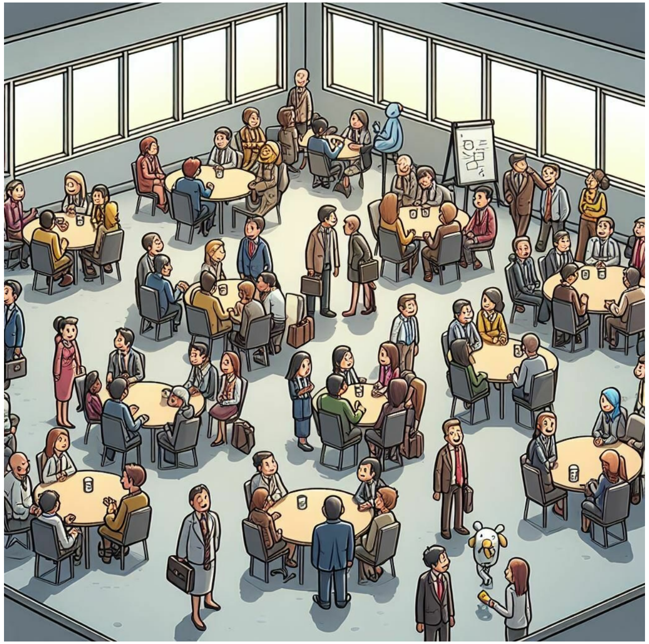
- Snapshot representation
- A set of n nodes $\mathcal{V}$
- A set of temporal edges

$$(i , j , t)$$

first node $i \in \mathcal{V}$

second node $j \in \mathcal{V}$

interaction timestamp



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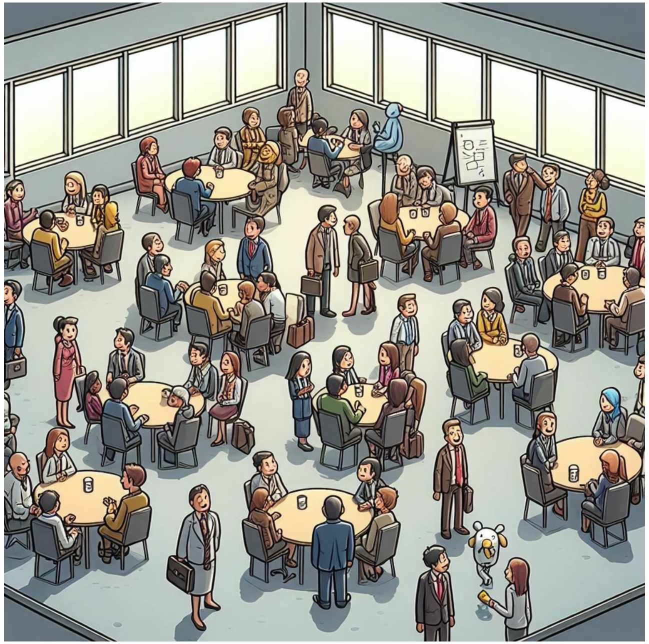

Temporal graphs can encode complex dynamic relations between entities

Question

Can we define a
<u>distance</u> *to compare*
temporal graphs?

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Only few related works:

- Bail et al (2023)
- Froese et al (2020),
- Zhanet al (2021)

Desiderata

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- Usual distance properties

1. non negativity
2. separation axiom
3. symmetry
4. triangle inequality

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  topological → how the nodes are connected

  temporal → when they get connected

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- Capture topological and temporal structure of the interactions

how the nodes are connected

when they get connected

how they are intertwined

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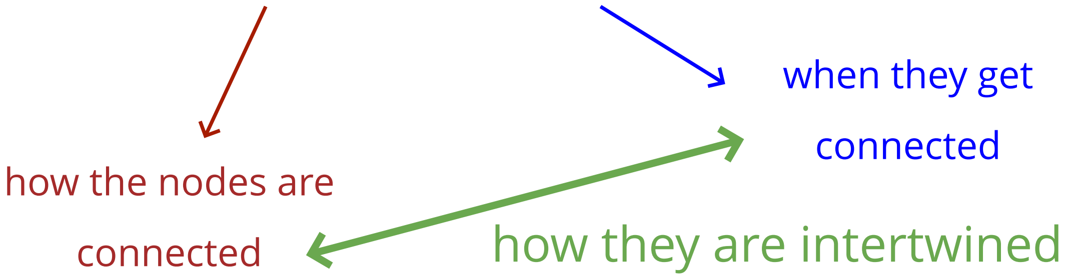

- Can compare graphs with different observation times T

(Un)matched graphs

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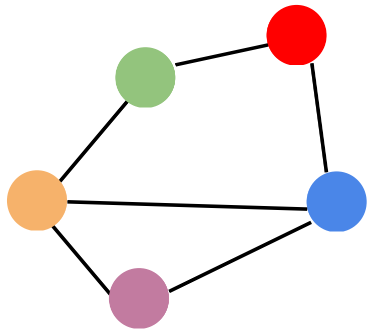
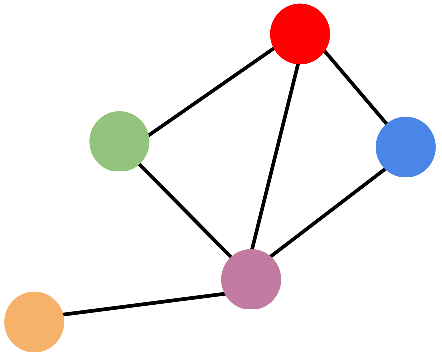
Matched graphs:

a **known** bijection between nodes





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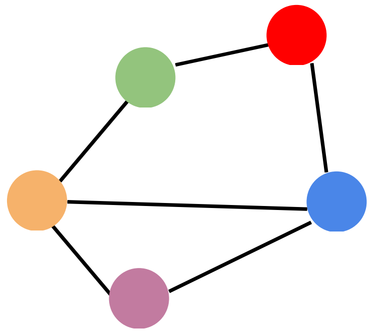

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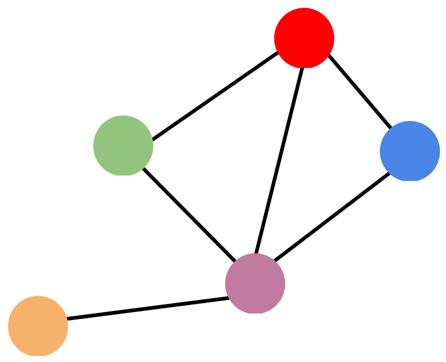

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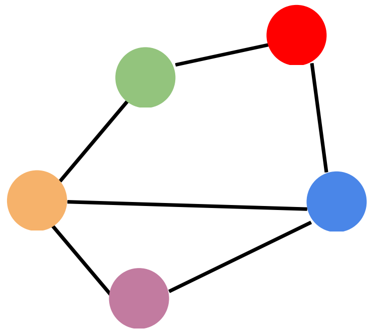

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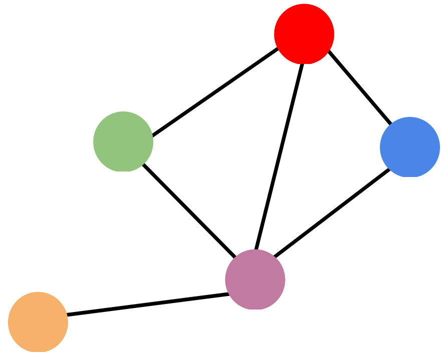
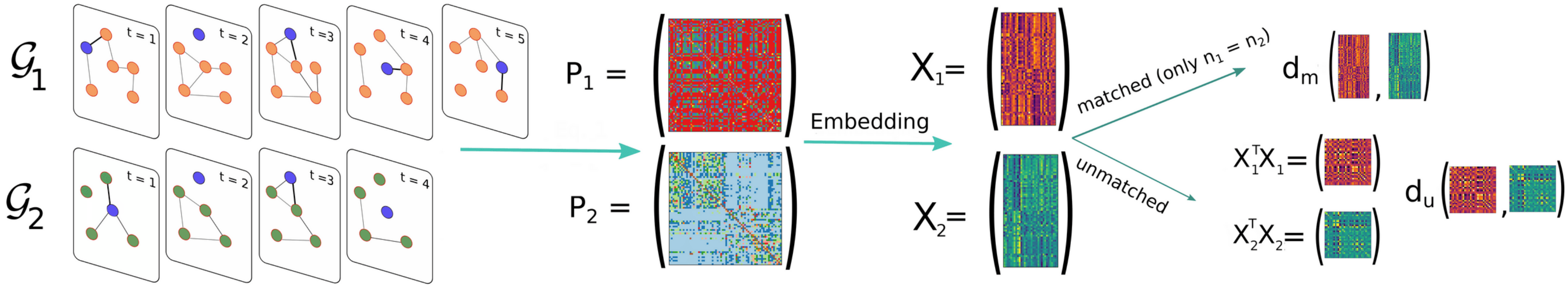


Two distances to handle both cases

Distance definition

The strategy



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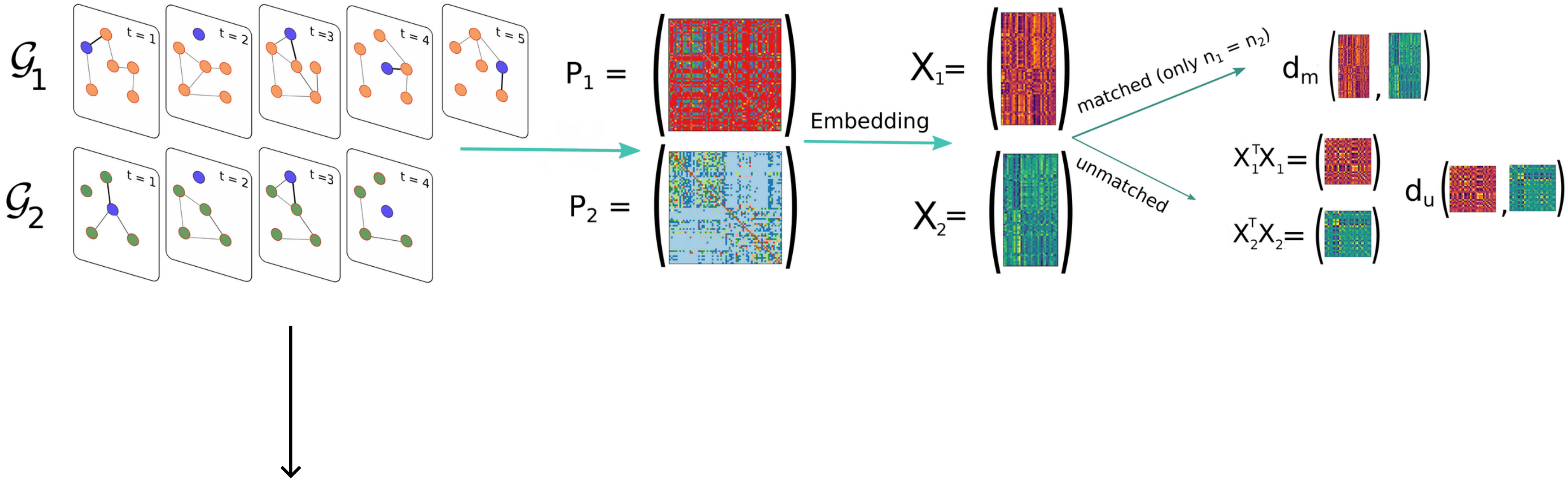

input graphs

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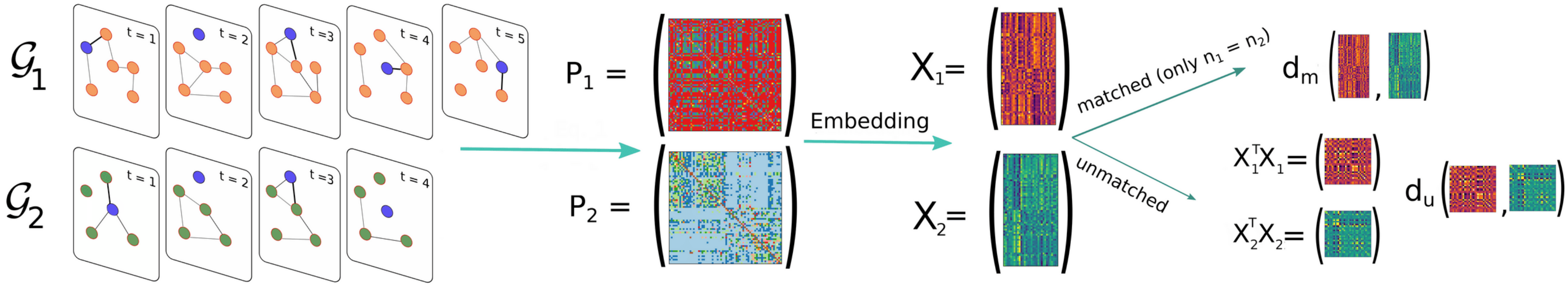

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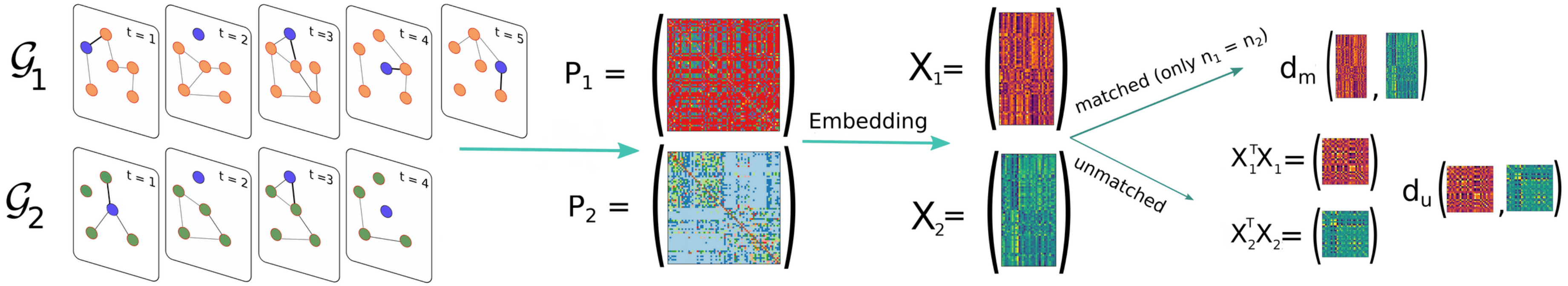

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embedding in vector space

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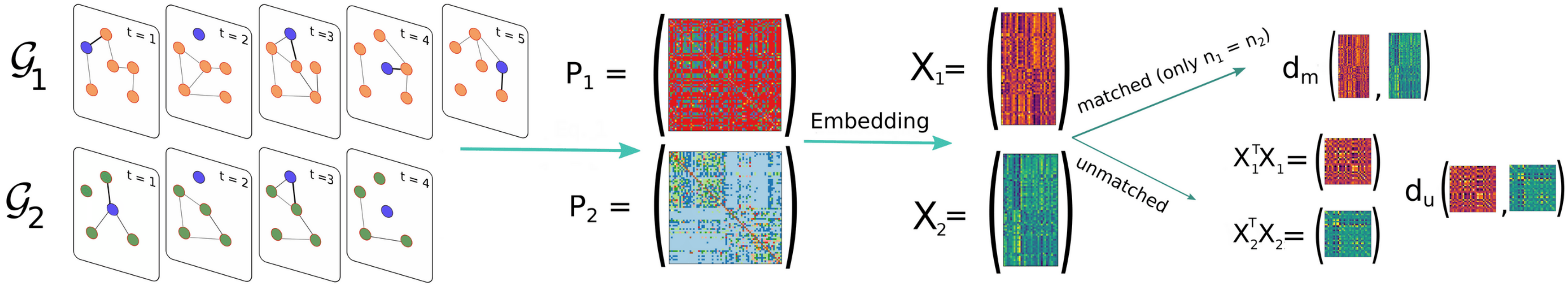

input graphs

transition probability matrix

embedding in vector space

distance in embedded space

Transition probability matrix

$$P = \frac{1}{T}\sum_{\tau=1}^{T} L_\tau L_{\tau-1}\cdots L_2 L_1$$

- T : total number of snapshots
- L_τ: row-normalized instantaneous adjaceny matrix with self-loops

P_{ij} *: limiting probability to go from i to j with time-respecting random walks*

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Depends on tological and temporal network structure

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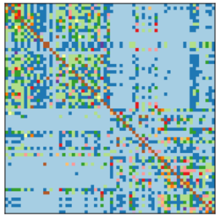
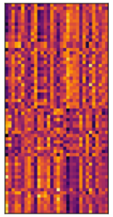
Embedding



$\boldsymbol{P} \in \mathbb{R}^{n \times n}$



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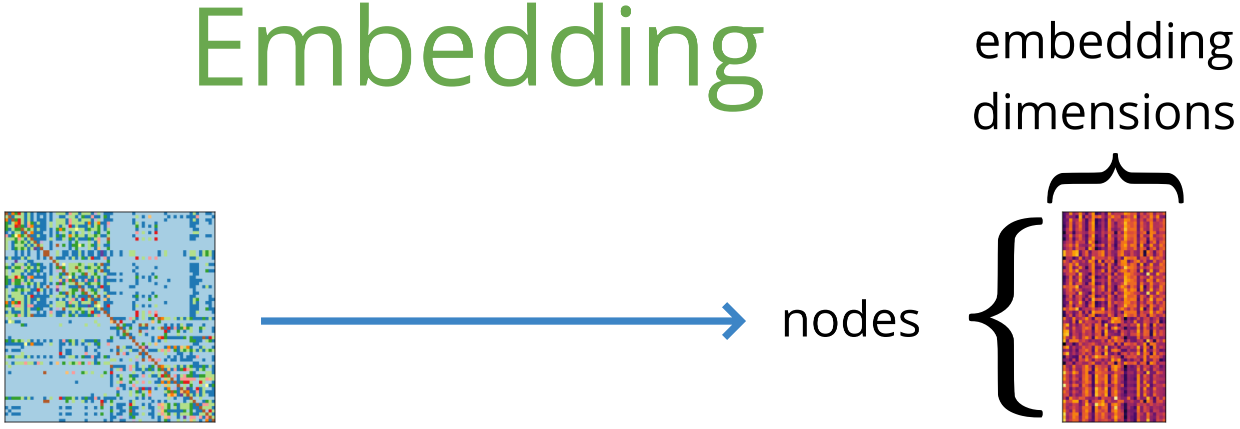

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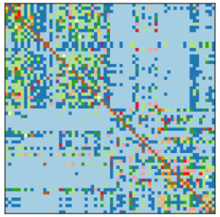
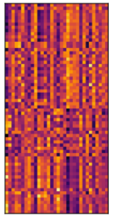


embedding dimensions

nodes



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$$\mathcal{L}(X) = -\sum_{i,j \in \mathcal{V}} \left(P_{ij} - \frac{1}{n} \right) X_i^T X_j - \log \underbrace{\left(\sum_{k \in \mathcal{V}} e^{X_i^T X_k} \right)}_{Z_i}$$

Embedding

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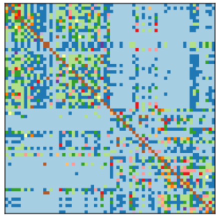

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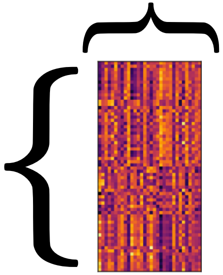

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Embedding

EDRep algorithm

- Fast
- Low dependence on d
- No need to compute P explicitely

https://arxiv.org/abs/2303.17475

embedding dimensions

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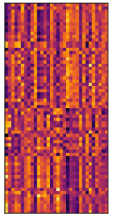

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Matched graphs

$$d_{\mathrm{m}} = \overbrace{\|X_1 X_1^T - X_2 X_2^T\|_{\mathrm{F}}}^{\text{expensive}}$$

Entry-wise comparison of similarity between node pairs

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$$= \underbrace{\sqrt{\|X_1^T X_1\|_{\mathrm{F}} + \|X_2^T X_2\|_{\mathrm{F}} - 2\|X_1^T X_2\|_{\mathrm{F}}}}_{\text{cheap}}$$

Entry-wise comparison of similarity between node pairs

Unmatched graphs

$$d_{\mathrm{u}} = \left\| \lambda\left(\frac{X_1^T X_1}{n_1}\right) - \lambda\left(\frac{X_2^T X_2}{n_2}\right) \right\|_2$$

$\lambda \in \mathbb{R}^d$: set of ordered eigenvalues

- *Invariant under node permutations*
- *Independent of graph size*

Evaluation

Method 1

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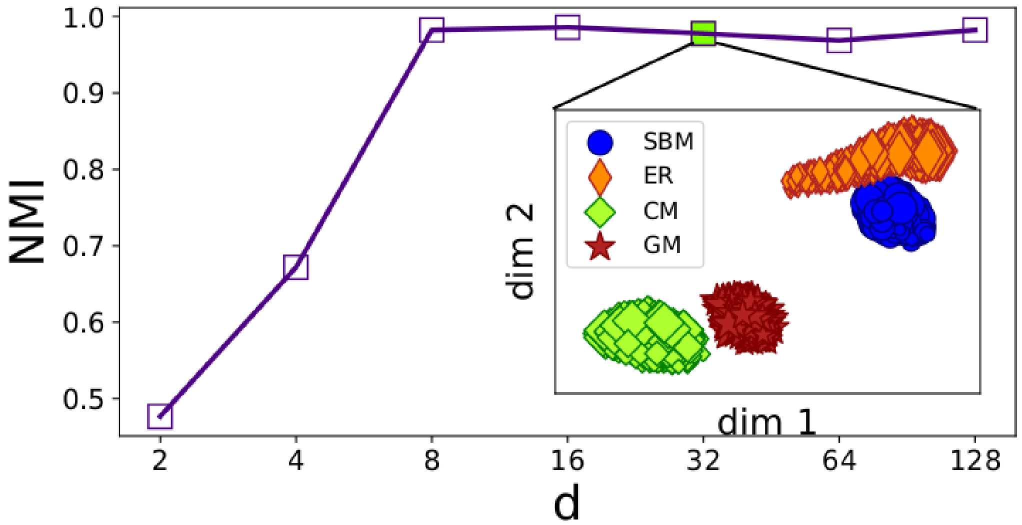

Reconstruction accuracy as function of embedding dimension

Method 2

Randomizations of real graphs

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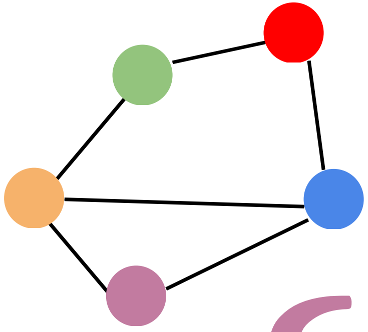
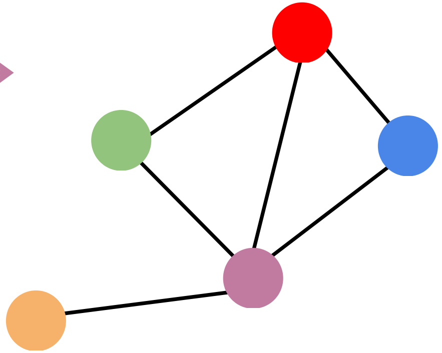
Gauvin et al, *Randomized Reference Models for Temporal Networks*

Randomizations of real graphs



Randomization preserving Q



Q preserved

- *Random:* # temporal edges
- *Random delta:* interaction duration distribution
- *Active snapshot:* node activity state, # of edges at t
- *Time:* aggregated graph
- *Sequence:* structure of each snapshot
- *Weighted degree:* # of interactions per node

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Distinguishing randomizations means to be sensitive to Q

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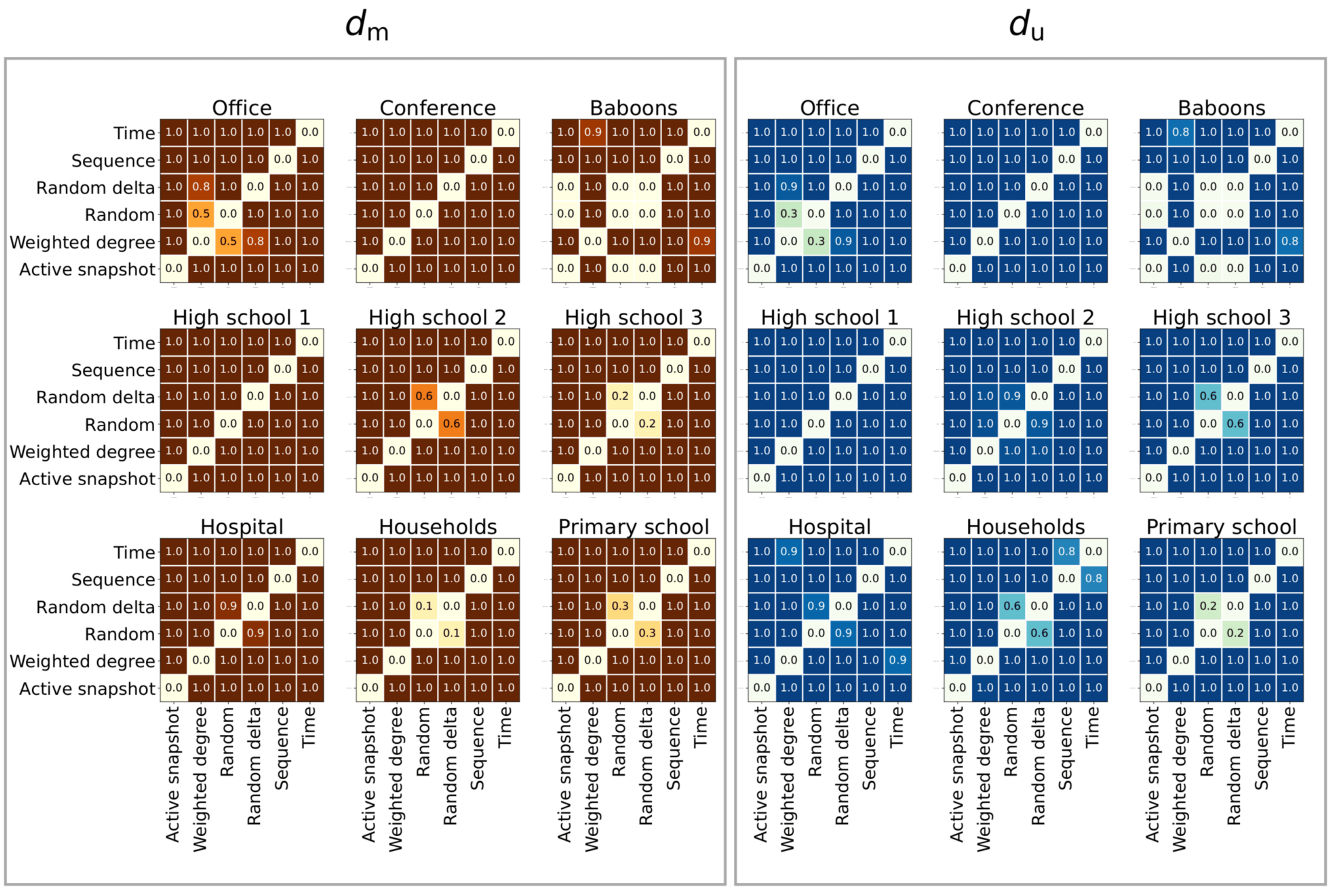

We can distinguish all shufflings pairs for almost all graphs

Conclusion

Contribution

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- Both **matched** and **unmacthed** cases
- Disciminate (synthetic, empirical) temporal networks with complex topological and temporal structures

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Outlook

- Data analysis
- Evaluate generative model

THANK YOU

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fondation BOTNAR

EPFL COVID-19 Real Time Epidemiology I-DAIR Pathfinder

periscope

European Union's Horizon 2020 No. 101016233

Fondazione CRT

Lagrange project

PAPER

CODE

arXiv:2401.12843