Spectral clustering in sparse heterogeneous networks

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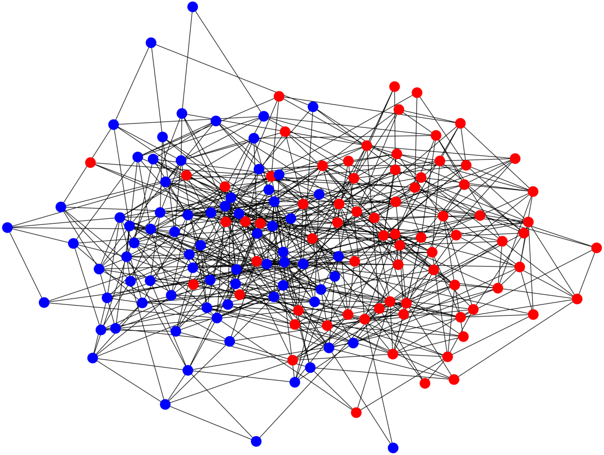
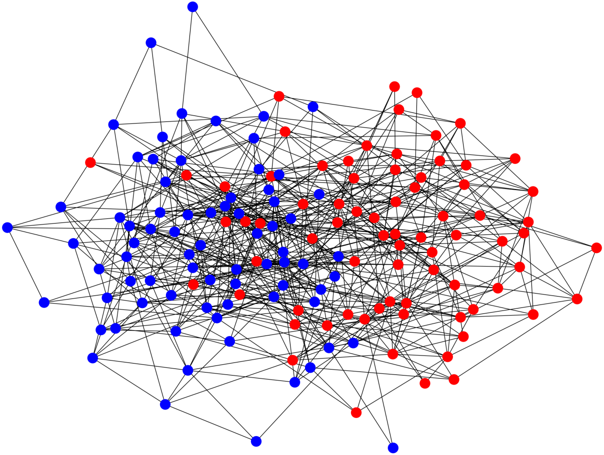
What and why



What and why

The solution...



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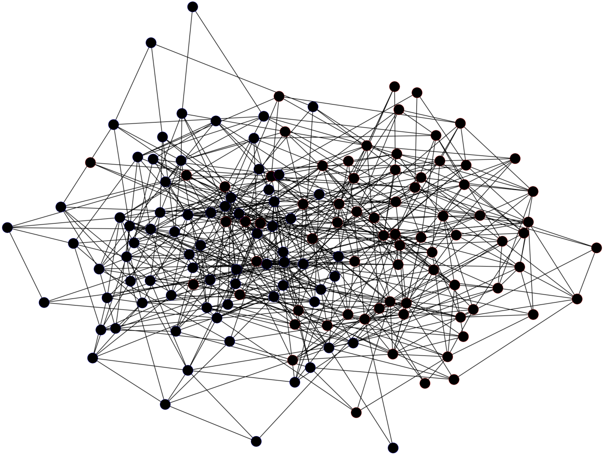

The spectral techniques

Information inside the eigenvectors

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UNSUPERVISED

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FAST — TRACTABLE — UNSUPERVISED

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Dense regime

$d \sim n$

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Average degree

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Size of the network

Example

- A : adjacency matrix, $A_{ij} = 1$ if (ij) are connected, zero else

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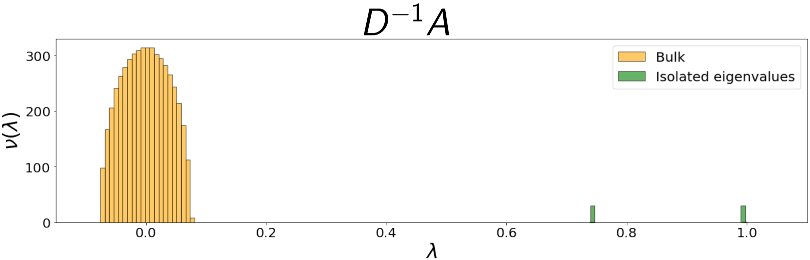

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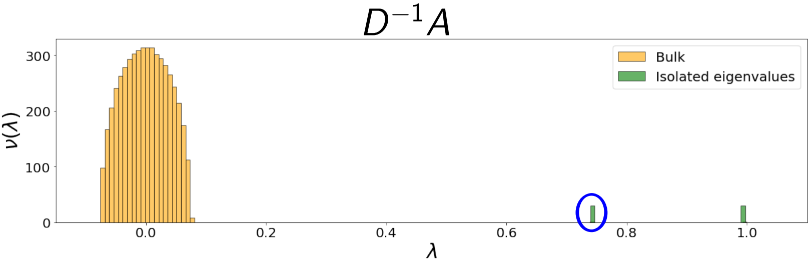

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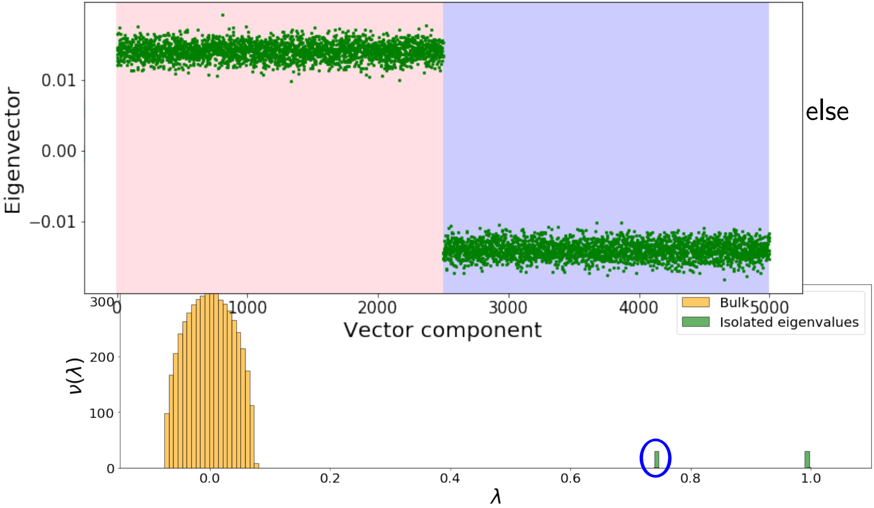
Eigenvector
0.01
0.00
−0.01
0
1000
2000
3000
4000
5000
Vector component
else
300
200
100
0
$\nu(\lambda)$
0.0
0.2
0.4
0.6
0.8
1.0
λ
Bulk
Isolated eigenvalues

Sparsity

Real networks are sparse

$$d \approx const$$

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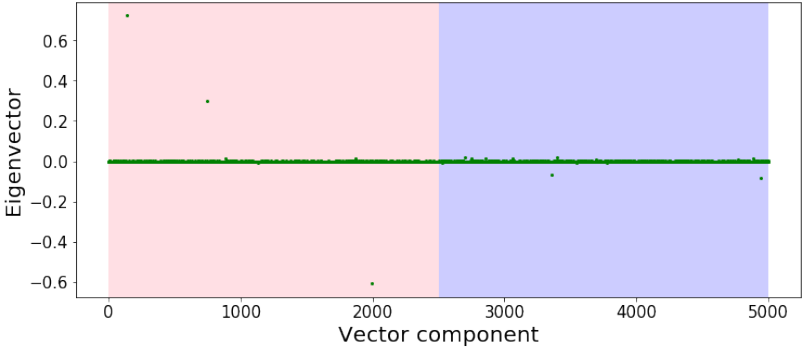

Two solutions

- Non-backtracking matrix[1] $B \in \{0,1\}^{2|\mathcal{E}|\times 2|\mathcal{E}|}$
  $B_{(ij),(kl)} = \delta_{jk}(1-\delta_{il})$

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  $H_r = (r^2 - 1)I_n + D - rA, \; r \in \mathbb{R}$

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Only for homogeneous degree distribution

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The DC-SBM

$$\mathbb{P}(A_{ij} = 1 | \sigma_i, \sigma_j, q_i, q_j) = q_i q_j \frac{C(\sigma_i, \sigma_j)}{n}$$

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Detectability threshold[3] :

$$\alpha := \frac{c_{\rm in} - c_{\rm out}}{\sqrt{c}} \geq \frac{2}{\sqrt{\Phi}}$$

[3] Gulikers *et al.* An impossibility result for reconstruction in the degree-corrected stochastic block model, The Annals of Applied Probability 2018

Goal

$$H_r = (r^2 - 1)I_n + D - rA, \quad r \in \mathbb{R}$$

Second smallest eigenvector of H_r for:

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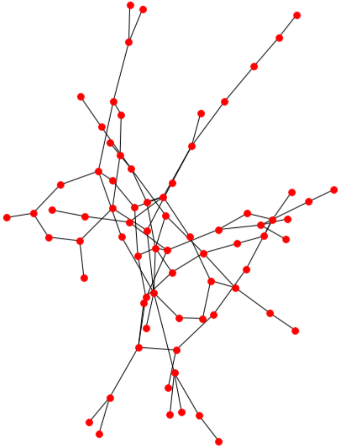
Pick the good value of r to:

- Retrieve $\boldsymbol{\sigma}$ regardless of $\boldsymbol{q}$

Tree like approximation



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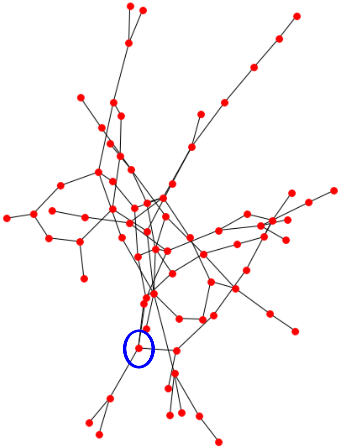

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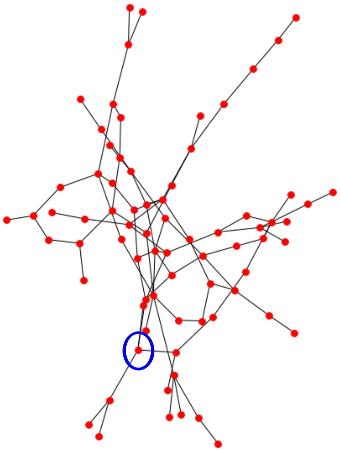

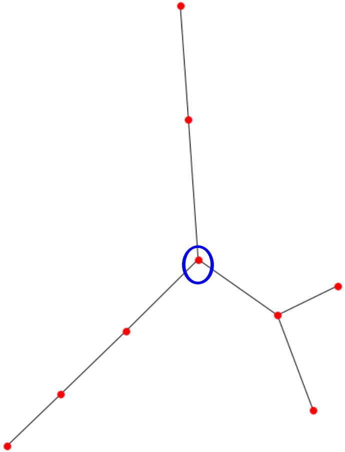

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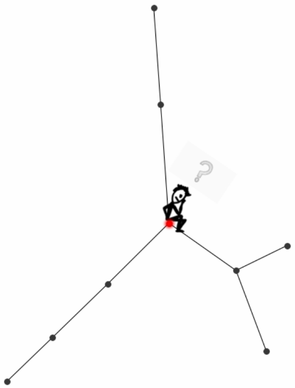

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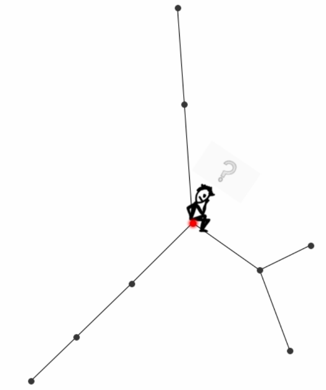
$$\mathbb{P}(\sigma_i = \sigma_j | A_{ij} = 1) = \frac{c_{\rm in}}{c_{\rm in} + c_{\rm out}} \qquad \mathbb{P}(\sigma_i \neq \sigma_j | A_{ij} = 1) = \frac{c_{\rm out}}{c_{\rm in} + c_{\rm out}}$$



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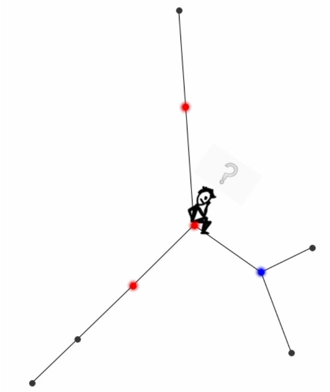

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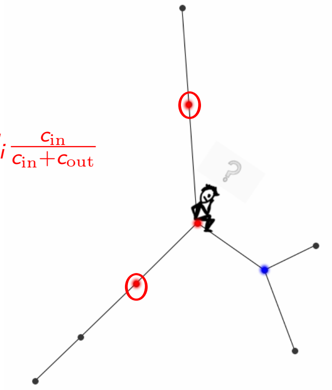

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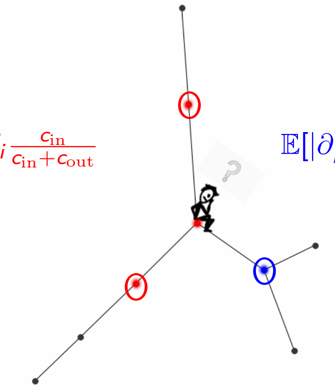

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$$r_{\rm opt} = \zeta \equiv \frac{c_{\rm in}+c_{\rm out}}{c_{\rm in}-c_{\rm out}} \equiv \zeta_\alpha = \frac{2\sqrt{c}}{\alpha}$$

From heuristics arguments...

$$Ov = 2\left(\frac{1}{n}\sum_{i=1}^{n}\delta_{\sigma_i,\tilde{\sigma}_i} - \frac{1}{2}\right)$$

Explicit expression of the overlap

$$\mathbb{E}[\mathrm{Ov}] \simeq \frac{1}{n}\sum_{i=1}^{n}\mathrm{erf}\left[\sqrt{\frac{\alpha^2 d_i}{8c-2\alpha^2}\left(\frac{c\Phi-\zeta_\alpha^2}{c\Phi-1}\right)}\right]$$

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$$\lim_{c_{\mathrm{out}}\to 0} 8c - 2\alpha^2 = 0$$

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$$\lim_{\alpha\to\alpha_c} c\Phi - \zeta_\alpha^2 = 0$$

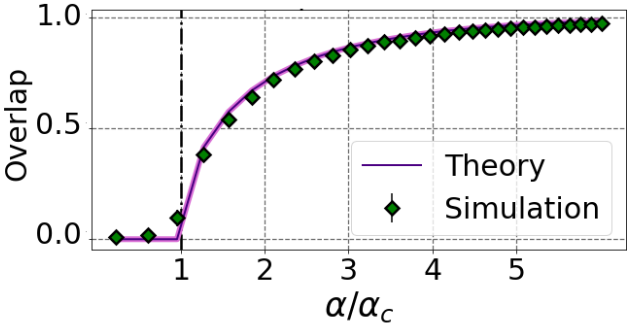

Figure: Simulated versus theoretical overlap for q_i distributed according to a power law. Average over 10 realizations. Both figures : The following parameters were used: $n = 5000$, $c_{\mathrm{out}} = 8$, $c_{\mathrm{in}} = 9 \to 61$.

How to estimate $r_{\rm opt} = \zeta_\alpha$?

From linearisation of BP

$$B\boldsymbol{w} = \zeta_\alpha \boldsymbol{w}$$

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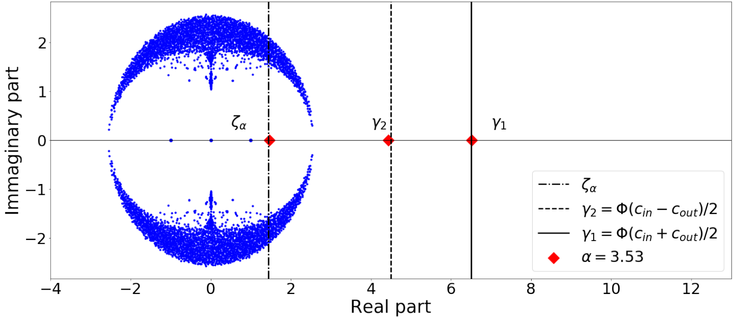

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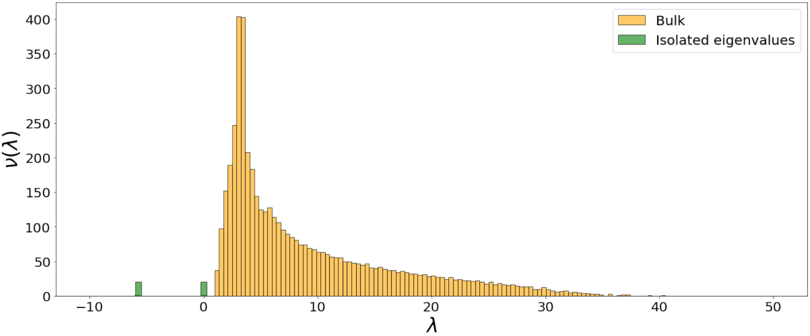

Spectrum of H_{ζ_α}

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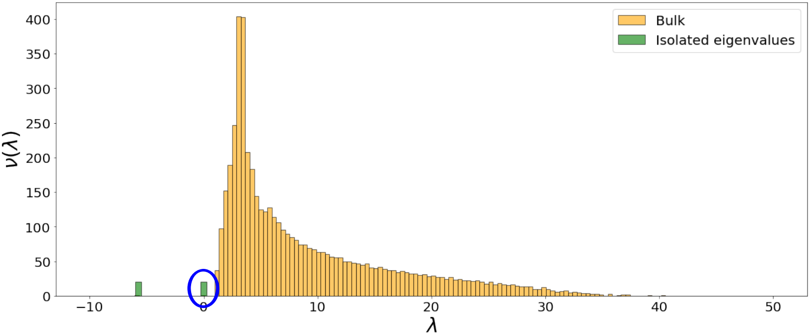
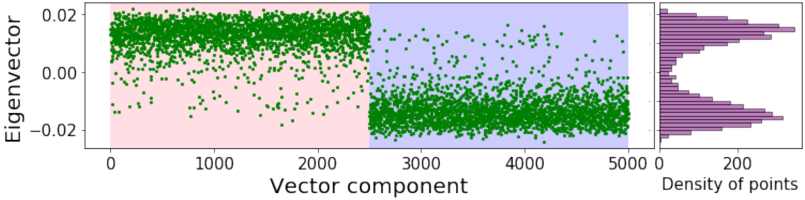


Spectrum of H_{ζ_α}

The eigenvector

Optimal value $r = \zeta_\alpha = \frac{c_{\rm in}+c_{\rm out}}{c_{\rm in}-c_{\rm out}}$



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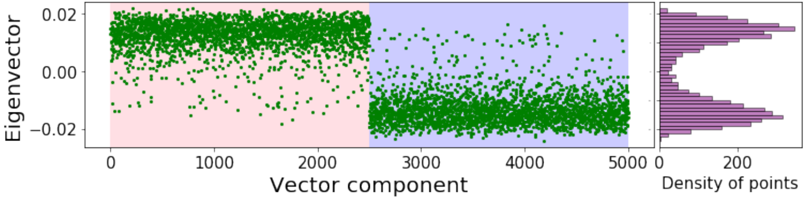
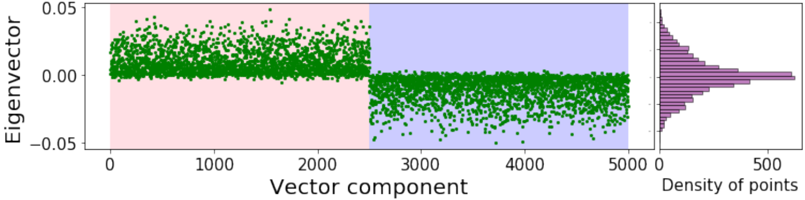


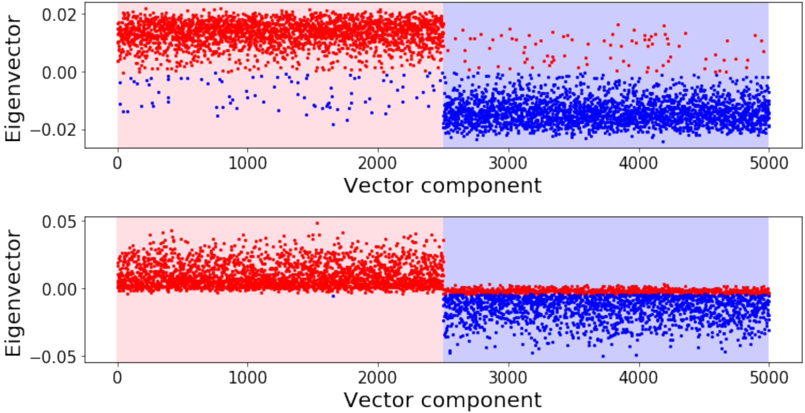
Initially proposed[4] value $r = \frac{\sum_i d_i^2}{\sum_i d_i}$



[4] Saade *et al.*, Spectral clustering of graphs with the Bethe-Hessian, NIPS 2014.

k - means

Optimal value $r = \zeta_\alpha = \frac{c_{\rm in}+c_{\rm out}}{c_{\rm in}-c_{\rm out}}$



The overlap

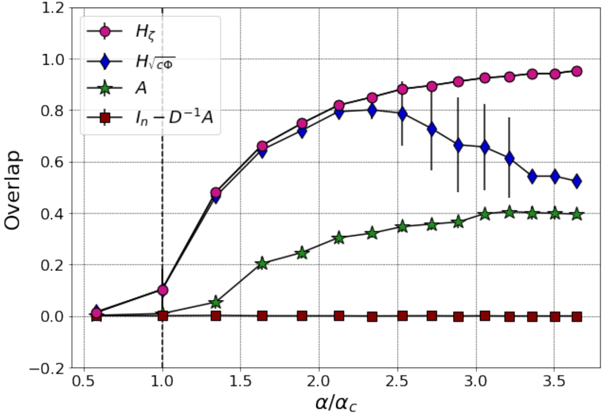


Figure: Overlap as a function of α/α_c for a power law degree distribution

More than two classes

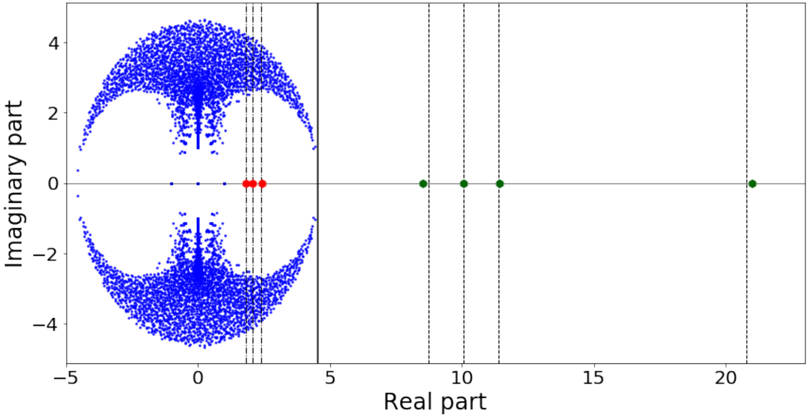


Figure: Spectrum of B for 4 classes

Future steps

- Theoretical support for our findings

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- Algorithmic optimization (eigencounts techniques)

Thank you!